

Invariance Groups in Classical and Quantum Mechanics

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To Professor KONRAD BLEULER on the occasion of his 60th birthday

This is a report on some new relations and analogies between classical mechanics and quantum mechanics which arise out of the work of Kostant and Souriau. Topics treated are i) the role of symmetry groups; ii) the notion of elementary system and the role of Casimir invariants; iii) energy levels; iv) quantisation in terms of geometric data on the classical phase space. Some applications are described.

1. Invariance Groups

In recent years new ideas have helped to clarify the relations and analogies between symmetry principles in classical mechanics, on the one hand, and quantum mechanics on the other hand. In this article I would like to describe some of these ideas.

The phase space M of a classical finite dimensional system with n degrees of freedom is a $2n$ -dimensional differentiable manifold. The Poisson bracket structure on M derives from a skew-symmetric second order covariant tensor ω which is locally expressible as $\sum_{i=1}^n dp_i \wedge dq_i$ in terms of local coordinates $q_1, \dots, q_n, p_1, \dots, p_n$. Equivalently we may say that ω is a skew-symmetric non-singular metric tensor satisfying $d\omega = 0$. Such a tensor ω is called a *symplectic form* on M , and the pair (M, ω) is called a *symplectic manifold*. A group G is called an *invariance group* of the system if an action of G on M is given such that the 2-form ω is invariant under the transformations of G , (such transformations are canonical transformations of M).

The phase space of a quantum mechanical system, on the other hand, is the set \hat{H} of rays (1-dimensional subspaces) of a complex Hilbert space H . The *transition probability* between two rays is defined to be the absolute value of the scalar product of unit vectors generating those rays. A group G is called an *invariance group* of the system if an action of G on \hat{H} is given which leaves the transition probability invariant.

2. Elementary Systems

A quantum mechanical system with phase space \hat{H} is regarded as *elementary* in the context of an in-

variance group G if the action of G on \hat{H} is irreducible (i.e. no proper closed subspaces of \hat{H} invariant under G). For example the elementary relativistically invariant quantum mechanical systems are associated with irreducible actions of the Poincaré group, as emphasized by Wigner.

A classical mechanical system with phase space M is, analogously, defined to be *elementary*, in the context of an invariance group G , if the action of G on M is transitive (i.e. no proper subsets of M invariant under G). This was suggested by BACRY³.

For the classification of elementary systems, both classical and quantum, with a prescribed Lie group G as invariance group, one uses the Casimir invariants of G . These are polynomials on the Lie algebra of G which are invariant under the adjoint action of G .

For quantum systems this is very well known since by Schur's lemma the Casimir invariants give operators on H which act by scalar multiplication in an irreducible representation. For classical systems an analogous, but less well known, situation holds. Here the Casimir invariants define a family of functions on the dual of the Lie algebra of G , and the level sets of this family are symplectic manifolds, the symplectic form in each case being derived from the Lie algebra of G . All elementary classical systems arise essentially in this manner. This was observed by KOSTANT³ and SOURIAU⁴ following work of KIRILLOV⁵. Similar ideas arise in the work of ANDRIÉ and BLEULER⁶.

Elementary classical relativistic systems were classified in this way by SOURIAU⁴ and later independently by ARENS⁷ and RENOARD⁸. Souriau also considered elementary Galilean systems, and RAWNSLEY⁹ has classified elementary de Sitter systems.



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3. Energy Levels

In quantum mechanics the eigenspace of the Hamiltonian eigenvalue E is itself a Hilbert space, called the *quantum mechanical energy level* with energy E . In classical mechanics we have an analogous concept. Here the set of all classical trajectories in $2n$ -dimensional phase space having energy E is a $(2n-2)$ -dimensional symplectic manifold, called the *classical energy level* with energy E .

For example, each of the classical energy levels for the Kepler problem of a particle of mass m in a $-K/r$ potential is a 4-dimensional symplectic manifold. It is in fact diffeomorphic to the product $S^2 \times S^2$ of two 2-spheres of radius mK , and $(2m)^{3/2} K \sqrt{-E}$ times the symplectic form is equal¹⁰ to the sum of the area elements of the two spheres.

4. Geometric Quantisation and the Kostant Functor

The problem of geometric quantisation is to give a prescription whereby one constructs a quantum phase space \hat{H} solely in terms of geometric data on an initial classical phase space M . The construction should be a natural one (functorial) in the sense that the presence of an invariance group at the classical level M should be preserved when we pass to the quantum level \hat{H} . It should also, when applied to the phase space of a free particle, yield the usual canonical quantisation.

Kostant has succeeded in obtaining such a prescription by requiring an additional structure, called a *polarisation*, on the classical phase space. By a polarisation of a symplectic manifold (M, ω) is meant an integrable field F of maximal isotropic (relative to ω) vector subspaces of the complex tangent spaces of M . The triple (M, ω, F) is then called a polarised symplectic manifold. In addition one requires that the 2-form ω should have integral periods, this is a 2-dimensional version of the Bohr-Sommerfeld conditions^{10, 11} and if it is satisfied we say that the symplectic manifold is *quantisable*.

The Kostant procedure gives, roughly speaking, a functor from the category of polarised quantisable symplectic manifolds [representing classical phase spaces] to the category of Hilbert ray spaces [representing quantum phase spaces]. This Kostant functor, as we shall call it, is described in some detail in

CARMONA¹² and in RENOARD⁸. An informal description may also be found in Reference¹³. In the following section we shall simply state the results obtained when the Kostant functor is applied to a few fundamental classical systems.

5. Examples

a) Free Relativistic Particle

The phase space of a classical free relativistic particle of positive mass m and spin zero is a 6-dimensional symplectic manifold (diffeomorphic to R^6) on which the Poincaré group acts transitively. It is quantisable for all values of m , and it carries a unique relativistically invariant polarisation. The Kostant functor, applied to this polarised quantisable symplectic manifold together with the action of the Poincaré group on it, yields the mass m and spin zero irreducible unitary ray representation of the Poincaré group.

For non-zero spin and positive mass an analogous situation holds, the principal difference being that the classical phase space is now 8-dimensional and is diffeomorphic to $R^6 \times S^2$ (where S^2 denotes the 2-dimensional sphere) and is quantisable only for half-integer spin.

For zero mass and spin s the classical phase space is 6-dimensional and is quantisable only for half-integer spin. The Kostant functor yields the zero mass spin s irreducible unitary ray representation of the Poincaré group.

Thus in all these cases the Kostant functor transforms the elementary classical relativistic particle into the corresponding elementary quantum mechanical relativistic particle. See Ref.⁸ for more details.

b) Energy Levels of the Hydrogen Atom

The 4-dimensional rotation group $SO(4)$ acts transitively on each negative energy level $S^2 \times S^2$ of the Kepler problem, as introduced in Section 3, and leaves the symplectic form invariant. Thus each energy level is an elementary $SO(4)$ -invariant classical system. It is quantisable only for energy $E = -2\pi^2 m K^2 / N^2$ where N is integer. It carries a certain $SO(4)$ -invariant polarisation, which is the only one yielding a non-trivial representation of $SO(4)$ under the Kostant functor. This representation is irreducible and N^2 -dimensional and thus may be associated with the quantum energy level with

energy $-2\pi^2 m K^2/h^2 N^2$. Thus the Kostant functor maps the elementary $SO(4)$ -invariant classical en-

energy level with energy E to the elementary $SO(4)$ -invariant quantum energy level with energy E/h^2 .

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- ² H. BACRY, I. A. S. preprint, Princeton 1966.
- ³ B. KOSTANT, Quantisation and unitary representations, Lectures in Modern Analysis and Applications II (ed. C. T. TAAM). Springer Lecture Notes in Mathematics **170**, 1970.
- ⁴ J.-M. SOURIAU, Structures des Systèmes Dynamiques, Dunod, Paris 1970.
- ⁵ A. A. KIRILLOV, Unitary representations of nilpotent Lie groups, Russian Math. Surveys **17**, No. 4, 53 [1962].
- ⁶ M. ANDRIÉ and K. BLEULER, Invarianzgruppen der klassischen Probleme der Mechanik I, II, Comm. Physico-Math. Soc. Sci. Fenn. **34**, **36** [1969].
- ⁷ R. ARENS, Classical Lorentz invariant particles, J. Math. Physics **12**, 2415 [1971].
- ⁸ P. RENOUARD, Variétés Symplectiques et Quantification, These, Orsay 1969.
- ⁹ J. RAWNSLEY, Some applications of quantisation, Thesis, Oxford 1972.
- ¹⁰ D. J. SIMMS, Equivalence of Bohr-Sommerfeld, Kostant-Souriau, and Pauli quantisation of the Kepler problem. Proceedings of colloquium on Group Theoretical Methods in Physics, C. N. R. S. Marseille 1972.
- ¹¹ N. E. HURT, A classification theory for quantisable dynamical systems, Reports on Math. Physics **2**, 211 [1971].
- ¹² J. CARMONA, Représentations des groupes de Lie d'après B. KOSTANT, Prepublications de la Recherche Cooperative sur Programme No. 25, vol. 9 (1969) CNRS.
- ¹³ D. J. SIMMS, Symplectic Manifolds, Quantisation, and Group Representations. Proceedings of conference on General Relativity and Mathematical Methods in Field Theory, Margaretenhöhe 1971.